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A Matrix Analysis of Processes Involving Particle Assemblies

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A MATRIX ANALYSIS OF PROCESSES INVOLVING PARTICLE ASSEMBLIES

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[Plate 1]

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The processes analyzed in this paper are the size reduction and size classification of particle assemblies. Particle size distributions are described by vectors, and alterations to size distributions during breakage processes are described by matrices multiplying the vectors. The matrix approximation gives an adequate representation of the processes studied, and the manipulation of the matrices is easy and flexible.

The breakage of a particle assembly is thought of as two processes. In the first, the machine breaking the particles is said to select for breakage a proportion of the particles, and the remaining particles are unbroken. To discover a function or matrix which describes the process of selection is to understand how the machine operates. In the second process, the particles selected are broken in a regular way; the proportions of particles of each size formed by the breakage are described by a breakage function or a breakage matrix. The analysis of breakage is in this way given convenient mathematical form. These matrices depend on the characteristics of the machine and on the nature of the particle assembly.

After breaking the particles, crushing and grinding machines frequently pass the product assemblies to a classifier from which the larger particles are returned, mixed with fresh material, to the grinding zone. The analysis is extended to the description of such circuits.

The experimental work reported concerns the breaking of coal particles in a new grinding machine, ball mills, shatter tests and a beater mill. The selection functions derived throw light on the operation of these machines. Coal breakage has been studied since it is an important field of application, and because coal is typical in breakage of homogeneous rocks. For each of the machines examined and for each particle size, a single breakage function has sufficed to describe the product of breakage: $[1-\exp(-z)]/[1-\exp(-1)]$ is the proportion of the product smaller than a fraction z of the original particle size.

1. Introduction

Regularities have often been observed in the size distributions of products from particle-breaking machines such as ball mills, rod mills, jaw crushers and swing-hammer mills. The breaking of homogeneous rocks, for instance limestones and dolomitic ores, quartz and coal, affords the simplest examples. Similar regularities are obtained under laboratory conditions, for example in shatter, grindability and compression tests. Study of the breakage of particle assemblies has nevertheless been confused and attempts to give a unified treatment of the subject have failed to produce an analysis of the operation of the simplest machines. The technology by which required products are made (for example pulverized fuel or materials suitable for briquetting) is therefore crude and relies largely on rule of thumb.

This paper proposes an analysis of the breakage processes of machines or of involuntary breakage (for example during the transportation of the particles). The experimental work reported concerns the breakage of coal only, since in this field application is important and immediate. However, coal is typical in breakage of homogeneous rocks, and the methods of analysis suggested are therefore quite general. The methods could be extended to the breakage of heterogeneous particles, and to the analysis of any process which changes the distribution of a characteristic in an assembly.

The treatment of particles in the grinding zones of machines cannot be observed directly, but information about the breakage may be deduced from the size distributions of the particles entering and leaving the grinding zones. When a particle passes through a grinding zone, it may remain intact or it may be broken in one of a number of different ways. The strains set up in a particle during its passage through the zone determine whether it breaks and, if broken, the size distribution of the product. These strains and the strength of the particle depend on many factors, the details of which are not usually known. The grinding of a whole assembly of particles is therefore too complex to analyze fully, but an analysis can be made if a simplified model is adopted.

The approximations used in this paper may be summarized as follows. Only one type of breakage is considered to take place, in which the size distribution of the product of breakage is described by the function given below and called the breakage function. For each particle broken this function gives the probability that the product contains particles of each size; when a large number of particles break, the same function gives the overall distribution of their product. Only a proportion of the particles of each size suffers this breakage and the remainder passes through the grinding zone unbroken. The probability of breakage depends generally on particle size, and is given for each size by a function called the selection function.

The characteristics of this mathematical model depend generally on the breakage function and selection function, but adequate descriptions of the breakage processes observed have been obtained with a single breakage function. The model is altered to agree with different circumstances by suitable choice of selection functions.

Epstein (1948) first explicitly paired the concepts of a breakage function and of the probability of breakage. Epstein deduced under general conditions the product distribution when the number of repeated breakages was large; an asymptotic relation may not be relevant to the passage of coal through a grinding zone in which the number of repeated breakages is small.

Previous searches for a 'law' of coal breakage have not always distinguished the idea of selection, and so cannot be used to determine a suitable breakage function. Two requirements of the function are that particles of all sizes may occur in a product of breakage from the initial particle size to the smallest size measured (in the experiments reported below, from 3 in. to about $100\,\mu$), and that no particle in the product may be larger than the original particle.

We have chosen as the breakage function

$$B(x,y) = [1-\exp(-x/y)]/[1-\exp(-1)] \quad (0<x\leqslant y).$$

Here $B(x,y)$ is the proportion by weight of the product whose size is less than x. The size of the original particle is y. 'Size' is taken as that length determined by sieving, that is, by the nominal or experimentally determined sieve aperture. According to this function, the products of breakage of two particles differ by a geometrical factor, the ratio of the sizes of the original particles.

This breakage function differs from previous laws of breakage (e.g. Bennett 1936; Brown 1941) in that the distribution is determined by the initial particle size, and not by a measured characteristic of the product. Its form resembles that of Rosin & Rammler (1933; see also Mittag 1953).

Other breakage functions could replace $B(x,y)$; the treatment of breakage given in this paper does not depend on the particular breakage function used. Even for the experiments described below, the function used may not give quite the best agreement between calculated and experimental distributions. Future developments in the physics of breakage may lead to more appropriate breakage and selection functions.

In the next section the breakage and selection functions are used in the analysis of processes which alter the size distributions of particle assemblies. Practical analysis of such processes requires a mathematical technique with the following properties: it must conveniently represent distributions and the processes which change distributions; it must extend to the description of associated processes (for example, closed circuit grinding); it must lead to simple numerical calculations and use measurements which can be made easily.

The matrix treatment which is introduced might have been presented in an abstract way which could be specialized for particular applications: we have chosen to use terms which describe coal breakage because this presentation seems clearest. The treatment is applied to data from some breakage processes in § 3.

2. Matrix analysis of breakage mechanisms

A batch of coal, consisting of particles of different sizes, is usually characterized by a cumulative function $F(y)$, which gives the proportion by weight of the batch below each size y, the sizes being properly defined. This function may be taken to be differentiable, and its slope at each point, $f(y)$, represents the proportion by weight with which the particles of each size occur. We call $\mathrm{d}F(y)/\mathrm{d}y = f(y)$ the frequency function. In practice we determine values of the cumulative function at a set of known points $a_0, a_1, \ldots, a_n$ that is, we find $F(a_0), F(a_1), \ldots, F(a_n)$. This we may do by a size analysis, in which, for example, the proportion by weight which passes a sieve of aperture a_i is regarded as $F(a_i)$.

The differences between successive values of $F(y)$, $F(a_i)-F(a_{i+1})$, e.g. the proportions passing a sieve of size a_i but retained on a sieve of size a_{i+1}, are the equivalent of a frequency function. These differences give the relative weights of particles between sizes a_i and a_{i+1}.

The set of values $F(a_0)-F(a_1), \ldots, F(a_{n-1})-F(a_n)$ we now arrange as a column vector, and this vector contains all the information we have on the size distribution. We write the vector $\{f_1, \ldots, f_n\}$ and call it the frequency vector $\mathbf{f}$. If a_1 is larger than the largest size in the batch, and if $f_{n+1} = F(a_n)$, i.e. the proportion of size less than a_n, then

$$f_1 + \ldots + f_n + f_{n+1} = 1.$$

To the frequency vector corresponds a cumulative vector with $n+1$ elements,

$$\{1, f_2 + \ldots + f_{n+1}, \ldots, f_{n+1}\}.$$

Its ith element is $F(a_i)$.

Suppose now the size distribution of a batch of coal is changed by a breakage process. Suppose also the process has a functional form which we write $b(x, y)$; this is, for each feed size y, the frequency function in x of the product. Weighting this function by the proportion of particles of each size, we obtain

$$p_1(x) = \int b(x, y)\, \mathrm{d}F(y) = \int b(x, y)\, f(y)\, \mathrm{d}y$$

for the frequency function of the product of a single cycle of breakage operating on all particles in the batch, the integration being over the range of y.

If this product is again submitted to the process of breakage, the product of the second cycle will be described by the frequency function

$$p_2(z) = \int b(z, x)\, p_1(x)\, \mathrm{d}x = \iint b(z, x)\, b(x, y)\, f(y)\, \mathrm{d}y\, \mathrm{d}x,$$

and in general to describe the product of several cycles of breakage we require a transformation of the feed distribution consisting of repeated integrals.

The evaluation of these repeated integrals and the solution of integral equations can be troublesome. We therefore investigate the equivalent process involving vectors.

It is well known (see, for instance, Jeffreys & Jeffreys 1946, p. 152) that such integrals may be regarded as the limiting cases of matrix multiplication. We have already noted that the feed distribution is often known in vector form; it is natural to take similar points of subdivision for the matrix equivalent of the breakage function.

Let b_{ij} be the proportion of a typical particle between a_{j-1} and a_j before breakage which falls between a_{i-1} and a_i after breakage $(i, j = 1, \ldots, n)$. If $\mathbf{B}$ is the breakage matrix with elements b_{ij}, and

$$\mathbf{p}_1 = \mathbf{Bf},$$

then $\mathbf{p}_1$ is a vector whose ith element is

$$p_i = b_{i1} f_1 + \ldots + b_{in} f_n \quad (i = 1, \ldots, n),$$

i.e. the proportion after breakage between a_{i-1} and a_i.

Therefore $\mathbf{p}_1$ is the frequency vector describing the batch after a single cycle of breakage. The product of two cycles of breakage will be described by

$$\mathbf{p}_2 = \mathbf{Bp}_1 = \mathbf{B}^2\mathbf{f}$$

and so on.*

* Suppose now we border $\mathbf{B}$ with an extra row and an extra column; the jth element of the new row is the proportion of a typical particle between a_{j-1} and a_j before breakage which has size less than a_n after breakage, the new column is

$$\{0, 0, \ldots, 0, 1\}.$$

Then the bordered matrix is a *transition matrix* (Feller 1950) which describes fully the change in size distribution on breakage. The borders are omitted in the present treatment since $\mathbf{B}$ contains all the information in the bordered matrix and has certain useful symmetric properties.

Similarly, if a breakage function and product distribution are given, and from them the feed distribution is to be determined, we have an integral equation to solve. Its solution in matrix terms can be written down immediately: it involves the inversion of the matrix **B**, and is

$$\mathbf{f} = \mathbf{B}^{-1}\mathbf{p}_1.$$

In general, the manipulation of matrix equivalents is likely to present fewer difficulties than the original forms.

It is clearly unnecessary to limit the treatment to differentiable cumulative functions, since for discrete or mixed distributions the proportion $F(a_i)-F(a_{i+1})$ is still defined, and the matrix approximation can be carried through. The integrals to which it approximates are now Stieltjes integrals.

When matrices are thus used to represent functions, they will only do so usefully on two conditions: first, that they give sufficiently accurate results, and secondly, that they can be manipulated easily. The first condition will usually be satisfied if n is sufficiently large, and this would seem to violate the second condition. The results required may or may not justify the labour and expense involved in handling large matrices. But in some of the problems we have considered, the matrices which arise are of a special type, which can be manipulated without great difficulty, and which are amenable to the calculation of products and inverses. That the first condition is also satisfied is shown by the agreement of experimental results with the predictions of the theory.

The accuracy obtainable using quite small matrices is also demonstrated by the following calculations. Suppose the feed cumulative function is

$$F(y,h) = y^h \quad (0<h<1),$$

and the cumulative function describing breakage is

$$[1-\exp(-x/y)]/[1-\exp(-1)].$$

Then the cumulative function describing the product of one cycle of breakage may be found explicitly:

$$P(x,h) = [1-e^{-x}+x^h G(x,h)]/[1-\exp(-1)],$$

where

$$G(x,h) = \int_x^1 e^{-u}u^{-h}\,du,$$

the difference of two incomplete gamma functions.

Similarly, the matrix **B** equivalent to this breakage function may be written down, and the product **p** obtained. The calculation is outlined in the next section.

In table 1 we give values of $P(x,h)$ for various x and h, and also the values of the matrix approximation using matrices of eight and of four elements. It will be seen that the matrix approximation, even with as few as four elements, is adequate for most practical purposes.

TABLE 1. EXACT VALUES OF AND MATRIX APPROXIMATIONS TO $P(x, h)$

	$h=0\cdot5$			$h=0\cdot75$			$h=0\cdot95$		
x	exact	8-element matrix	4-element matrix	exact	8-element matrix	4-element matrix	exact	8-element matrix	4-element matrix
1	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
$\frac{1}{2}$	0·940	0·940	0·942	0·911	0·916	0·920	0·894	0·897	0·905
$\frac{1}{4}$	0·802	0·802	0·804	0·731	0·734	0·738	0·685	0·687	0·696
$\frac{1}{8}$	0·642	0·642	0·644	0·536	0·540	0·543	0·476	0·477	0·484
$\frac{1}{16}$	0·494	0·493	0·495	0·373	0·375	0·377	0·312	0·310	0·315

Since the size of a particle cannot be increased by breakage, $b_{ij} = 0$ for $i<j$, i.e. $\mathbf{B}$ is a lower triangular matrix. In processes other than breakage this simplification does not always hold good, and in a growth process the matrix would be upper triangular.

A further simplification is sometimes possible. Let the breakage function, here $b(x, y)$, be of the form $b(x/y)$, i.e. particles of every size are broken in the same way and the product depends only on a scale factor, the size y of the original particle. Then if the critical sizes are chosen in geometric progression (the largest size being taken as 1)

$$a_i = a^i \quad (a<1;\ i = 0, 1, \ldots, n),$$

it follows that

$$b_{ij} = b_{i-j+1,\,1} \quad (i \geqslant j).$$

The matrix $\mathbf{B}$ is in this case completely determined by its first column:

$$\mathbf{B} = \begin{pmatrix} b_1 & 0 & 0 & \ldots & 0 \\ b_2 & b_1 & 0 & \ldots & 0 \\ b_3 & b_2 & b_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & b_{n-1} & b_{n-2} & \ldots & b_1 \end{pmatrix}.$$

This may also be written

$$\mathbf{B} = b_1\mathbf{I} + b_2\mathbf{Y} + \ldots + b_n\mathbf{Y}^n,$$

where $\mathbf{Y}$ is an n^2 matrix with 1 immediately below the main diagonal and 0 elsewhere. Matrices of this type form a commutative ring with multiplicative identity (the unit matrix, $\mathbf{I}$). The product of two such matrices $\mathbf{B}$ and $\mathbf{C}$ has the first column

$$b_1c_1,\ b_1c_2 + b_2c_1,\ \ldots,\ b_1c_n + \ldots + b_nc_1.$$

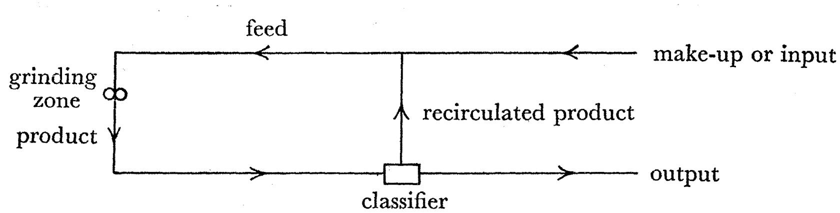
When b_1 is non-zero, $\mathbf{B}$ has a multiplicative inverse, also in the ring. If $\mathbf{C} = \mathbf{B}^{-1}$, the first column of $\mathbf{C}$ is given by solving in succession Wronski's relations (Aitken 1939)

$$\begin{aligned} b_1c_1 &= 1, \\ b_1c_2 + b_2c_1 &= 0, \\ &\vdots \\ b_1c_n + \ldots + b_nc_1 &= 0. \end{aligned}$$

It is because of the simplicity of numerical work with these matrices that their practical application is successful. For example, the work involved in calculating the product of two n^2 matrices in the ring increases with $\frac{1}{2}n^2$, and not with n^3 as for the product of general matrices.

Now that we have represented breakage in a convenient way, we turn to the description of a complete mechanism of breakage, or of a machine for breakage. We continue to describe the mechanism of coal breakage, but the method is applicable to all machines and processes which alter the distribution of a characteristic.

The most general circuit for pulverizing or crushing coal is shown diagrammatically:



A mixture of feed and recirculated material is fed continuously into the grinding zone of a mill or pulverizer. From the mixture a selection is made of the particles, and the selected particles are broken. The mixture of broken and unbroken particles forms the product, which is classified, that is, divided into an 'undersize' which is the output required, and an 'oversize' which is returned. For steady-state conditions, recirculated material must have make-up material added to it at the same mass flow rate as the output is produced. The system above does not necessarily operate in the steady state. Further, the classifier may send all the product of a single passage through the grinding zone to output; this is 'open-circuit' grinding. Alternatively, all make-up may be supplied at the beginning of grinding, which is thereafter 'batch' grinding, i.e. the classifier recirculates all the product, which is returned again and again to the mill. All circuits intermediate to these extremes effect 'closed-circuit' grinding.

Suppose the feed to the mill is denoted by $\mathbf{f}$, and in the ith size range a proportion s_i is selected for breakage. Let $\mathbf{S}$ be the n^2 matrix with $s_1, \ldots, s_n$ on the main diagonal and 0 elsewhere. $\mathbf{S}$ may or may not depend on $\mathbf{f}$, and the elements of $\mathbf{S}$ may or may not vary with i. If

$$\mathbf{D} = \mathbf{BS} + (\mathbf{I} - \mathbf{S}), \tag{1}$$

then $\mathbf{D}$ describes the operation of the grinding zone.

Similarly, the classification process is described by an n^2 matrix $\mathbf{C}$ with elements $c_1, \ldots, c_n$ on the main diagonal and 0 elsewhere, where c_i is the proportion of the product in the ith size range retained for recirculation. It is necessary to consider also c_{n+1}, the proportion of the product of size below a_n which is recirculated; this is often zero.

Suppose the size distribution and mass flow rate of the make-up material are represented by the vector $\mathbf{m}$ and M respectively, and that we use $\mathbf{f}$ and F to represent the feed, $\mathbf{p}$ and P for the product, $\mathbf{r}$ and R for the recirculated coal, $\mathbf{q}$ and Q for the output.

Then the operation of the machine is described completely by equations (1) to (9)

$$F = M + R_1, \tag{2}$$

$$P = F, \tag{3}$$

$$P = Q + R_2, \tag{4}$$

$$F\mathbf{f} = M\mathbf{m} + R_1\mathbf{r}_1, \tag{5}$$

$$\mathbf{p} = \mathbf{Df}, \tag{6}$$

$$P\mathbf{p} = Q\mathbf{q} + R_2\mathbf{r}_2, \tag{7}$$

$$R_2\mathbf{r}_2 = P\mathbf{Cp}, \tag{8}$$

$$Rr_{n+1} = Pc_{n+1}p_{n+1}. \tag{9}$$

Here $R_1\mathbf{r}_1$ describes the recirculated material mixed with the feed, and $R_2\mathbf{r}_2$ that separated by the classifier. In the steady state these are identical and we deduce $M = Q$.

Since equations (5) to (8) are vector equations, each vector having n elements, there are $4(n+1)$ equations implied in (2) to (9). It will be seen that the system is physically determined when it is in a steady state and M, $\mathbf{m}$, $\mathbf{D}$, $\mathbf{C}$ and c_{n+1} are given, and there are then $4(n+1)$ unknowns: F, P, R, Q, $\mathbf{f}$, $\mathbf{p}$, $\mathbf{r}$ and $\mathbf{q}$. Inadequate description of the performance of pulverizers is partly due to the failure to formulate these equations and secure the appropriate data.

We consider the solution in two particular cases; in both we suppose $\mathbf{m}$, $\mathbf{B}$, $\mathbf{C}$ known and $c_{n+1} = 0$. First, we suppose open-circuit operation, i.e. $\mathbf{C} = 0$, and $\mathbf{q} = \mathbf{p}$ known, that is, we actually measure the product of one passage through the mill, and the method of selecting for breakage, $\mathbf{S}$, is to be found with certain restrictions. Secondly, we suppose closed-circuit operation, with $\mathbf{D}$ known, and solve for the whole system in a steady state. Solutions have been obtained in other cases.

For the first case, the selection for breakage in each size range may depend on several factors, for example, the largest size present, or some function of the actual particle sizes. Suppose $\mathbf{S}$ is an unknown linear function of known diagonal matrices

$$\mathbf{S} = \pi\mathbf{S}_1 + \omega\mathbf{S}_2 + \ldots,$$

where the $\mathbf{S}_1, \mathbf{S}_2, \ldots$ can be deduced from our knowledge of the machine but the proportions in which these factors enter, π, ω, ..., are unknown and have to be estimated. (The parameters $\pi, \omega, \ldots$ conceal the strength of the coal, which is usually unknown.) For example, if $\mathbf{S}_1 = \mathbf{I}$, $\mathbf{S}_2 = \ldots = 0$ the selection process is simply that a proportion π in each size range is broken. In another instance we might have

$$\begin{aligned} \mathbf{S}_1 &= (1, \ldots, 1, 0, \ldots, 0), \\ \mathbf{S}_2 &= (0, \ldots, 0, (a_i a_{i+1})^{\frac{1}{4}}, \ldots, (a_{n-1} a_n)^{\frac{1}{4}}), \\ \mathbf{S}_3 &= \ldots = 0, \end{aligned}$$

then a proportion π in the first i size ranges is broken, but the proportions in the smaller sizes vary both with ω and with the square root of the average size in each range.

Equations (1) and (6) may be written, in cases where $\mathbf{S}_3 = \ldots = 0$,

$$\mathbf{p} - \{\mathbf{B}(\pi\mathbf{S}_1 + \omega\mathbf{S}_2) + (\mathbf{I} - \pi\mathbf{S}_1 - \omega\mathbf{S}_2)\}\,\mathbf{f} = 0.$$

This equation is sufficiently general to exemplify the method of solution proposed. We write it

$$\boldsymbol{\alpha} - \pi\boldsymbol{\beta} - \omega\boldsymbol{\gamma} = 0, \tag{10}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are vectors of $(n+1)$ elements, the undersize in equation (6) being taken into account:

$$\boldsymbol{\alpha} = \begin{pmatrix} \mathbf{p} - \mathbf{f} \\ p_{n+1} - f_{n+1} \end{pmatrix},$$

$$\boldsymbol{\beta} = \begin{pmatrix} (\mathbf{B} - \mathbf{I})\,\mathbf{S}_1\mathbf{f} \\ \{b_{n+1}, b_{n+1} + b_n, \ldots, b_{n+1} + \ldots + b_2\}\,.\,\mathbf{S}_1\mathbf{f} \end{pmatrix},$$

and

$$\boldsymbol{\gamma} = \begin{pmatrix} (\mathbf{B} - \mathbf{I})\,\mathbf{S}_2\mathbf{f} \\ \{b_{n+1}, b_{n+1} + b_n, \ldots, b_{n+1} + \ldots + b_2\}\,.\,\mathbf{S}_2\mathbf{f} \end{pmatrix}.$$

The vector equation (10) represents $(n+1)$ linear equations, and we have only two unknowns, π and ω. Only n of these equations are independent, and generally they cannot be satisfied simultaneously; this is frequently due to errors of measurement. If we believe the errors have equal effect at each element of the vector in equation (10) it is appropriate to give a least-squares solution:

$$\left.\begin{aligned} \pi\boldsymbol{\beta}^2 + \omega\boldsymbol{\beta}\,.\,\boldsymbol{\gamma} &= \boldsymbol{\alpha}\,.\,\boldsymbol{\beta}, \\ \pi\boldsymbol{\beta}\,.\,\boldsymbol{\gamma} + \omega\boldsymbol{\gamma}^2 &= \boldsymbol{\alpha}\,.\,\boldsymbol{\gamma}. \end{aligned}\right\} \tag{11}$$

In other circumstances the errors in the cumulative vector corresponding to (10) may be thought of equal effect, or we may decide on a particular range of sizes to use in the estimation of π and ω. In either case least-squares equations corresponding to (11) may be written down and solved.

It often happens that $\mathbf{S}_1 = \mathbf{I}$ and $\mathbf{S}_2 = \ldots = 0$, i.e. that a proportion π of the particles in each size range is broken, and a proportion $(1-\pi)$ is left intact. This process is called a *cycle of π-breakage*. If the whole product of one cycle of π-breakage is returned for a second cycle, and this is repeated until the batch has passed ν times through a cycle of π-breakage, the complete process is called *ν cycles of π-breakage*. The simple concept of a cycle of π-breakage is sufficient to explain many practical breakage processes.

In the estimation of π after a single cycle of π-breakage $\boldsymbol{\gamma} = 0$ and (11) has solution

$$\pi = \boldsymbol{\alpha}\,.\,\boldsymbol{\beta}/\boldsymbol{\beta}^2. \tag{12}$$

If we define vectors $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$, with n elements each,

$$\left.\begin{aligned} \alpha'_j &= \sum_{i=j+1}^{n+1} \alpha_i, \\ \beta'_j &= \sum_{i=j+1}^{n+1} \beta_i \end{aligned}\right\} \quad (j = 1, \ldots, n)$$

then the cumulative vector equation corresponding to (10) is

$$\boldsymbol{\alpha}' - \pi\boldsymbol{\beta}' = 0,$$

with least-squares solution

$$\pi = \boldsymbol{\alpha}'\,.\,\boldsymbol{\beta}'/\boldsymbol{\beta}'^2. \tag{13}$$

Any subset of sizes may be used for estimation; examples are given in the next section.

More generally, equations (1) and (6) may be used to estimate the actual $\mathbf{B}$ matrix, provided sufficient data are available and reasonable assumptions (or measurements) can be made on $\mathbf{S}$. If the breakage function is known to be of the form $b(x/y)$, then there are only n elements in $\mathbf{B}$ to be estimated.

In the second case we obtain the solution for a steady state, when the make-up distribution and operations of the grinding zone and classifier are all known. We suppose in addition that $c_{n+1} = 0$, i.e. that all material under the smallest size is sent through the classifier to output. Taking the make-up rate as the unit of mass flow rate (i.e. $M = 1$) we deduce from (2) to (4) that

$$M = Q = 1 \quad \text{and} \quad F = P = R+1.$$

Since $r_{n+1} = 0$ by (9) we deduce from (5) that

$$(R+1)\,f_{n+1} = m_{n+1}. \tag{14}$$

The solution follows from (5), (6) and (8):

$$\mathbf{f} = \left(\frac{1}{R+1}\right)(\mathbf{I}-\mathbf{CD})^{-1}\,\mathbf{m}. \tag{15}$$

Summing the $n+1$ equations implied by (14) and (15) we obtain

$$1 = \left(\frac{1}{R+1}\right)\{m_{n+1} + \Sigma(\mathbf{I}-\mathbf{CD})^{-1}\,\mathbf{m}\}, \tag{16}$$

where the summation is over the n elements of the vector.

Thus to calculate the steady state the main work is to invert the matrix $(\mathbf{I}-\mathbf{CD})$. The value of R follows from (16) and $\mathbf{f}$ from (15). The other unknowns are easily found, e.g. the distribution of the output $\mathbf{q}$ is given by (7)

$$\mathbf{q} = (R+1)(\mathbf{I}-\mathbf{C})\,\mathbf{Df},$$

or as

$$\mathbf{q} = (\mathbf{I}-\mathbf{C})\,\mathbf{D}(\mathbf{I}-\mathbf{CD})^{-1}\,\mathbf{m}$$

which does not involve R.

The solution is particularly simple when the breakage process is the selection of a proportion π (independent of size) which is broken according to $\mathbf{B}$, and classification sends all over a certain size to be rebroken, all under that size to output. Taking this critical size as a_n, we have $\mathbf{C} = \mathbf{I}$ and $c_{n+1} = 0$. Then (15) reduces to

$$\mathbf{f} = (\mathbf{I}-\mathbf{B})^{-1}\,\mathbf{m}/\pi(R+1).$$

When m_{n+1} is small the relation between $(R+1)$ and $1/\pi$ is nearly linear, and the distributions of feed and output are nearly independent of π.

The equivalent analysis of recirculation using integrals of distribution functions involves integral equations soluble only numerically and with difficulty.

3. Some breakage mechanisms analyzed

In this section we apply the treatment which we have developed to actual breakage mechanisms. We find that the simplest idea, a cycle of π-breakage, is sufficient to explain some practical results; others require a more complex analysis.

We first translate our breakage function into practical matrix terms. Suppose the critical sizes are $1, a, \ldots, a^n$, 1 being the largest particle size present. The first column of $\mathbf{B}$ has elements which are the proportions of a typical particle between 1 and a which fall after breakage between the limits 1 and a, a and $a^2, \ldots, a^{n-1}$ and a^n. If the typical particle may be taken to have size $a^{\frac{1}{2}}$, these elements are

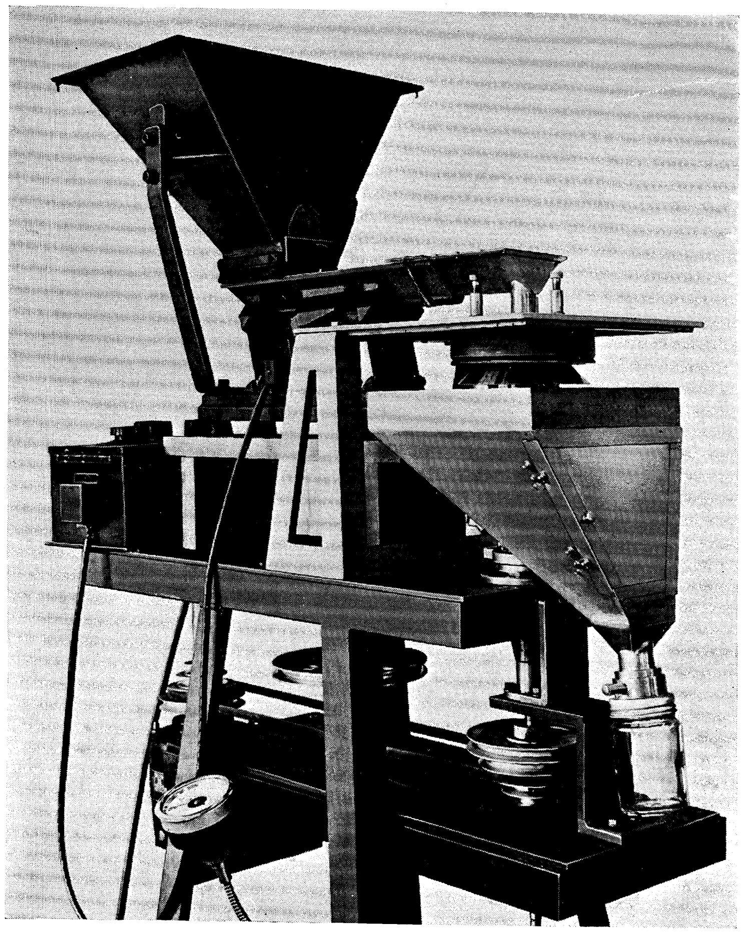
$$\begin{aligned}
b_1 &= [\exp(-a^{\frac{1}{2}})-\exp(-1)]/[1-\exp(-1)],\\
b_2 &= [\exp(-a^{\frac{3}{2}})-\exp(-a^{\frac{1}{2}})]/[1-\exp(-1)],\\
&\;\;\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots\\
b_n &= [\exp(-a^{\frac{1}{2}(2n-1)})-\exp(-a^{\frac{1}{2}(2n-3)})]/[1-\exp(-1)].
\end{aligned}$$

The proportion of size less than a^n is

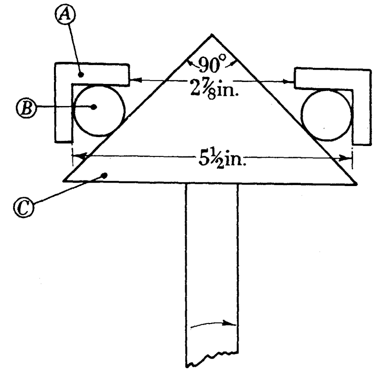
$$b_{n+1} = [1-\exp(-a^{\frac{1}{2}(2n-1)})]/[1-\exp(-1)].$$

The elements $b_1, b_2, \ldots, b_n$ are the numerical entries in the breakage matrix $\mathbf{B}$, which is an element of the ring defined in § 2.

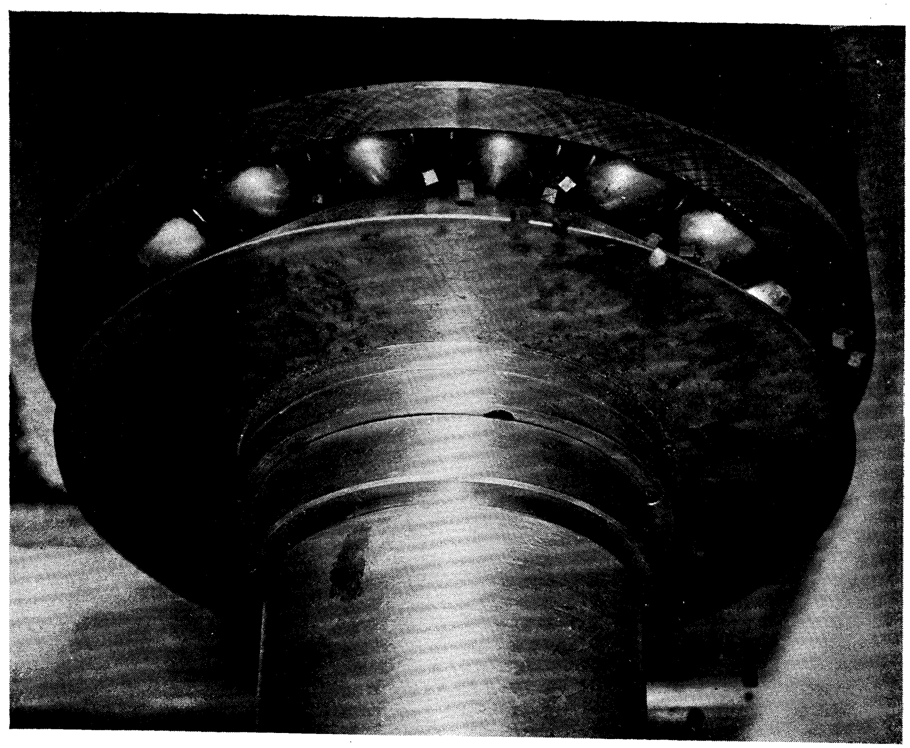
It is desirable to say what accuracy we require in our calculations. We have decided to use vectors of between six and nine elements, with a between 0·5 and 0·75; we have kept four decimal places in proportions (i.e. two decimals in percentages) and report three decimal places. This size of vector appears from table 1 to give sufficient accuracy; multiplication by matrices of this size is easily effected on a desk calculating machine. More important guides to the accuracy required are the two following points.



(a)



(b)



(c)

FIGURE 1. (a) General view of cone mill with cover lowered. (b) Section through cone mill. (c) High-speed flash photograph of $\frac{1}{4}$ in. cubes of coal passing through the grinding elements of the cone mill which rotated at 125 rev/min.

(Facing p. 109)

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Sieving provides a convenient but not always precise technique for determining the size analyses of coal samples. Imprecision arises both from sampling and from the sieving technique itself. In our work, sampling problems were not important since the entire product and feed of breakage processes was sieved (except in one ball-mill experiment). Consistent results can be obtained with a strictly defined technique; the application of the calibration technique of Carpenter & Deitz (1951) enabled a definition of effective size for each sieve between 16 and 170 mesh. We had no means of using their technique for larger sizes, and believe that inconsistencies in definition may exist for such sizes.

Since coals gain or lose moisture when the atmospheric vapour pressure varies, duplicate size analyses at any given sieve are in good agreement if the differences are less than 0·2 % of the total weight. Accurate distribution curves are most difficult to obtain with the finer meshes, since the small particles are most responsive to changes in moisture content and are most easily lost during handling. We have not used sieves finer than 150 mesh. The size analyses of duplicate samples obtained from separate grindings are in good agreement if the cumulative percentages passing any sieve differ by 1 % of the total sample weight.

We write throughout 'in good agreement' as a shorthand for 'are in good agreement as far as this preliminary investigation and current usage warrant', since in the technological applications envisaged, theory and practice are often (for practical purposes) considered to be in agreement when the predicted and actual percentages differ by even more than 1 % of the total sample weight. We nevertheless report the first decimal of percentages, since agreement of this order is sometimes attained.

Our experimental verification tests two assumptions. The first is that the matrix treatment, in a form amenable to calculations, gives useful approximations. The second is that the concept of selection, with the breakage function we have chosen, accounts for practical breakage processes. The simplest type of selection is π-breakage, we therefore first study cycles of π-breakage.

It is not obvious that any of the readily available grinding machines can be used to carry out one or more cycles of π-breakage, as this process implies that the probability of breakage is independent of particle size, that particles of every size break in the same way, and that the product of the breakage of a particle is not subjected within the cycle to further grinding. To satisfy these conditions a new mill has been built. This mill has been used to effect ν cycles of π-breakage on samples of three dissimilar coals, and this is the first experiment reported.

Three common breakage mechanisms are then discussed; a shatter test for determining the friability of coal, a ball mill and a beater mill.

The cone mill

The cone mill (Broadbent & Callcott 1954) shown in figure 1, plate 1, consists of a retaining ring (A) which maintains fourteen 1 in. diameter steel balls (B) in a horizontal track round a mild steel cone (C). The cone can be rotated at several rates. Coal particles are fed on to the apex of the cone by a Vibro-feeder at a sufficiently slow rate to ensure that the particles move down the face of the cone into the track of the balls, where breakage occurs. To ensure that all the crushed product is collected covers are fitted. It is not only the loss of coal which leads to a discrepancy in a material balance, but also the loss of moisture which results from size reduction. This discrepancy is minimized by using samples thoroughly air-dried and by carrying out size analyses immediately after grinding. The loss of weight in a single cycle did not exceed 0·3 %.

We consider that all particles passing a 7 mesh B.S. sieve (i.e. of a nominal diameter less than 2411 μ) are likely to behave in the same way in this mill, and that by sieving the product on a number of sieves between 7 and 150 mesh we obtain a satisfactory measurement of the size distribution.

It was convenient to commence a series of grinding operations with samples passing the 7- but retained on the 16-mesh sieves. The samples were prepared from larger coal by grinding and sieving, great care being taken to achieve closely similar samples within one batch of coal.

The chemical analyses of the coals used in this work are shown in table 2.

TABLE 2. PROPERTIES OF THE COALS USED IN THE EXPERIMENTS

	Gedling	Wyllie	Markham	Bilsthorpe	Chisnall Hall
proximate analysis:					
moisture (%), as received	10·3	0·9	6·9	6·5	2·9
ash (%), as received	3·4	3·7	4·4	6·8	16·1
volatile matter d.a.f. (%)	40·5	25·9	35·9	36·7	38·9
ultimate analysis:					
carbon (%), Parr's basis	81·4	89·5	83·8	81·9	84·0
hydrogen (%), Parr's basis	5·4	5·0	4·8	5·0	5·5

For all experimental work with the cone mill, we used vectors of nine elements, and so a 9×9 breakage matrix. The ratio of the critical sizes was taken as 0·747; the largest size being taken as 2411 μ, the remaining critical sizes were 1801, 1345, 1005, 751, 561, 419, 313, 234 and 175 μ. The first column of the breakage matrix is then

$$\{0{\cdot}0846,\ 0{\cdot}1629,\ 0{\cdot}1472,\ 0{\cdot}1267,\ 0{\cdot}1053,\ 0{\cdot}0852,\ 0{\cdot}0675,\ 0{\cdot}0529,\ 0{\cdot}0407\}.$$

Gedling: three cycles

The comparison of theory with experiment is now given in detail for three cycles of Gedling coal, to demonstrate the simplicity of the working. The sample, of 100 g weight, was passed one, two and three times at about 10 g/min through the cone mill rotating at 125 rev/min. The size distributions before the experiment and after each cycle were obtained by a sieve analysis. The percentage under each of the critical sizes was obtained from this analysis by graphical interpolation and is given in table 4. In this table the sizes are given in microns; they do not represent a particular dimension of the irregularly shaped particles, but that nominal dimension deduced from sieving. In the calculation made, size is dimensionless, being expressed as a fraction of the largest particle size, which we take to be 2411 μ corresponding to the 7-mesh sieve.

We suppose that the distributions obtained result from ν cycles ($\nu = 1, 2, 3$) of π-breakage, where π is to be estimated. Several criteria for the estimation of π may be used when feed and product distributions of a single cycle of breakage are given. The methods are set out in detail in table 3, in which we estimate π from the initial distribution $\mathbf{f}$ and the product of one cycle of breakage $\mathbf{p}_1$. Each of these is given as a frequency vector, e.g. between 1801 and 1345 μ there was 0·395 of the initial sample, and 0·364 of the product. In addition to the nine elements of $\mathbf{p}_1$ we give the proportion under the smallest size, 175 μ.

TABLE 3. GEDLING: ESTIMATION OF π

size (μ)	$\mathbf{f}$	$\mathbf{p}_1$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\alpha}'$	$\boldsymbol{\beta}'$
2411	0·390	0·3760	−0·0140	−0·3570	0	0
1801	0·395	0·3640	−0·0310	−0·2981	0·0140	0·3570
1345	0·215	0·2084	−0·0066	−0·0751	0·0450	0·6551
1005	0	0·0283	+0·0056	+0·1426	0·0516	0·7302
751	0	0·0056	+0·0056	+0·1228	0·0233	0·4876
561	0	0·0039	+0·0039	+0·1021	0·0177	0·4648
419	0	0·0032	+0·0032	+0·0826	0·0138	0·3627
313	0	0·0022	+0·0022	+0·0656	0·0106	0·2801
234	0	0·0018	+0·0018	+0·0513	0·0084	0·2145
175	0	(0·0066)	+0·0066	+0·1632	0·0066	0·1632

$$\sum_{i=1}^{10} \alpha_i \beta_i = 0{\cdot}021\,434, \qquad \sum_{i=1}^{10} \beta_i^2 = 0{\cdot}308\,184,$$

$$\sum_{i=1}^{9} \alpha_i' \beta_i' = 0{\cdot}102\,597, \qquad \sum_{i=1}^{9} \beta_i'^2 = 1{\cdot}826\,242,$$

$$\sum_{i=3}^{9} \alpha_i' \beta_i' = 0{\cdot}068\,120, \qquad \sum_{i=3}^{9} \beta_i'^2 = 1{\cdot}269\,637.$$

$$100\pi_1 = 3{\cdot}9, \qquad 100\pi_4 = 5{\cdot}6,$$
$$100\pi_2 = 4{\cdot}0, \qquad 100\pi_5 = 5{\cdot}4.$$
$$100\pi_3 = 7{\cdot}0,$$

TABLE 4. GEDLING: CUMULATIVE SIZE DISTRIBUTIONS (PERCENTAGES) OBSERVED AND CALCULATED ($\pi = 0{\cdot}042$)

size (μ)	feed obs.	$\nu=1$ obs.	$\nu=1$ calc.	$\nu=2$ obs.	$\nu=2$ calc.	$\nu=3$ obs.	$\nu=3$ calc.
2411	100·0	100·0	100·0	100·0	100·0	100·0	100·0
1801	61·0	62·4	62·5	66·0	63·9	66·8	65·3
1345	21·5	26·0	24·2	30·3	26·9	33·2	29·4
1005	0	5·2	3·0	8·3	6·0	10·8	8·9
751	0	2·3	2·5	4·4	4·9	6·5	7·2
561	0	1·8	1·9	3·2	3·9	4·7	5·8
419	0	1·4	1·5	2·6	3·0	3·7	4·6
313	0	1·1	1·2	2·0	2·4	2·9	3·6
234	0	0·8	0·9	1·6	1·8	2·3	2·8
175	0	0·7	0·7	1·2	1·4	1·8	2·1

The calculation of the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\alpha}'$, $\boldsymbol{\beta}'$ follows from the work in § 2, and the estimates π_1, π_2, π_3 and π_4 are then derived. These estimates minimize the sums of squares of differences in, respectively, the largest size range, the undersize, the frequency vector and the cumulative vector. For our purposes special interest attaches to the distribution of particles less than 1005 μ, and an estimate π_5 which minimizes the sum of squares of the differences between the last seven elements of the cumulative vectors (whereas π_4 makes use of all these elements) has been found particularly useful. Table 5 gives evidence that shows π_5 is probably the least variable of our estimates. The five estimates of table 3 are all of the same order.

In precisely the same way, estimates of π using $\mathbf{p}_1$ as feed and $\mathbf{p}_2$ as product, and $\mathbf{p}_2$ as feed and $\mathbf{p}_3$ as product, were obtained. The three values thus found for $100\pi_5$ were 5·4, 3·8 and 3·3; their arithmetic mean is 4·2, and we conclude that for Gedling coal under these conditions about 4·2 % of the particles are broken at each cycle, i.e. $\pi = 0{\cdot}042$. This apparent decrease in π has not been investigated.

With this value of π we calculate the breakage matrix $\pi\mathbf{B}+(1-\pi)\,\mathbf{I}$; its first column is

$$\{0{\cdot}9618,\ 0{\cdot}0068,\ 0{\cdot}0061,\ 0{\cdot}0053,\ 0{\cdot}0044,\ 0{\cdot}0036,\ 0{\cdot}0028,\ 0{\cdot}0022,\ 0{\cdot}0017\}.$$

This matrix is then multiplied once, twice and thrice into the feed vector **f**, giving the calculated distributions shown in cumulative form in table 4; e.g. while 61 % of the feed material was less than 1801 μ, after one passage through the cone mill 62·4 % was less than this size, and by calculation with 4·2 % breakage we expect 62·5 %. From this table it is clear that the matrix calculation of ν cycles of 4·2 % breakage agreed reasonably well with the experimental results. We discuss these results after further and more critical experiments substantiate this conclusion.

Markham: ten cycles

It is desirable to repeat the experiment just described with different coals and for larger values of π. Further, the agreement between theory and experiment will be more striking if the estimation of π does not involve all the data used in the comparison.

Two samples of Markham Main Hards, each of 100 g weight, were passed successively, one, two, ..., six and ten times at about 5 g/min through the cone mill rotating at 148 rev/min. Between each passage each sample was subjected to a sieve analysis, and the two size distributions at each value of ν were averaged. We knew, therefore, the cumulative distribution vectors for $\nu = 0, 1, 2, 3, 4, 5, 6$ and 10. From the first six cycles the various estimates of π were calculated (excluding π_1 which was found too variable to be reliable) and these are given in table 5. The estimate with the least variability is π_5, and its arithmetic mean is 0·067. This was taken as the value of π under these conditions.

TABLE 5. MARKHAM: ESTIMATION OF π BY π_2, π_3, π_4 AND π_5 FROM THE $(\nu-1)$TH TO THE νTH CYCLE

	$\nu=1$	$\nu=2$	$\nu=3$	$\nu=4$	$\nu=5$	$\nu=6$
$100\pi_2$	7·1	6·9	6·9	7·7	6·7	6·6
$100\pi_3$	6·7	6·6	8·2	10·5	8·0	9·2
$100\pi_4$	6·4	6·3	7·3	8·2	7·3	7·9
$100\pi_5$	6·5	6·1	6·8	7·1	6·9	7·0

TABLE 6. MARKHAM: CUMULATIVE SIZE DISTRIBUTIONS (PERCENTAGES) OBSERVED AND CALCULATED ($\pi=0{\cdot}067$)

size (μ)	feed obs.	$\nu=1$ obs.	$\nu=1$ calc.	$\nu=2$ obs.	$\nu=2$ calc.	$\nu=3$ obs.	$\nu=3$ calc.	$\nu=4$ obs.	$\nu=4$ calc.	$\nu=5$ obs.	$\nu=5$ calc.	$\nu=10$ obs.	$\nu=10$ calc.
2411	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0
1801	49·5	52·7	52·6	55·5	55·5	59·2	58·3	64·0	60·8	67·5	63·3	78·0	73·3
1345	17·0	21·0	21·6	25·5	25·9	30·5	30·0	35·1	33·8	40·3	37·5	58·0	52·9
1005	0	4·9	4·8	9·0	9·3	13·7	13·6	18·4	17·8	22·6	21·7	39·5	38·9
751	0	3·4	3·8	6·7	7·5	10·4	11·1	14·0	14·6	17·5	17·9	34·0	33·2
561	0	2·7	3·0	5·4	6·0	8·3	8·9	11·3	11·8	14·0	14·6	27·0	28·0
419	0	2·1	2·4	4·3	4·7	6·5	7·1	8·8	9·4	11·4	11·8	22·4	23·4
313	0	1·7	1·8	3·5	3·7	5·2	5·6	7·2	7·5	9·2	9·5	18·5	19·5
234	0	1·4	1·4	2·8	2·9	4·2	4·4	5·8	5·9	7·6	7·5	15·4	16·1
175	0	1·1	1·1	2·3	2·2	3·5	3·4	4·9	4·7	6·2	6·0	12·5	13·3

With this π, the cumulative distributions expected for the first five and the tenth cycle are given in table 6, where they can be compared with those obtained in the experiment.

In figure 2 the distributions observed and calculated are also shown graphically. The agreement, even at ten cycles, is very good, although the last experimental values were not used in the estimation of π.

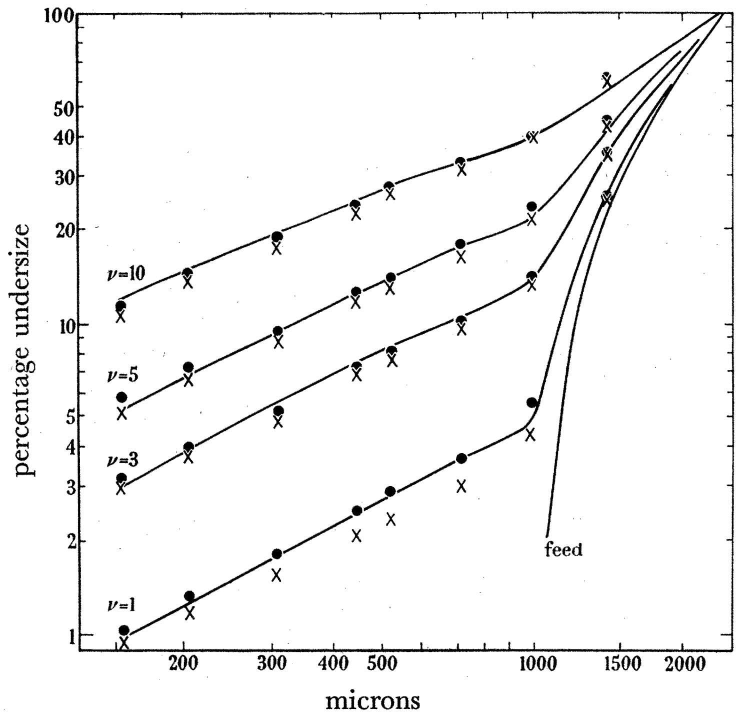


FIGURE 2. Cumulative size distributions: Markham. ——, calculated; ×, sample A; ●, sample B.

Wyllie: ten cycles

A single sample of this soft coal was treated in exactly the same way as that described for Markham. In table 7 we give values of π_5 calculated from the first five cycles of breakage. The arithmetic mean is 0·109. The expected and observed cumulative distributions are given in table 8. The agreement must again be considered very good.

TABLE 7. ESTIMATION OF π (WYLLIE FED AT 5 G/MIN)

ν	=	1	2	3	4	5
$100\pi_5$	=	12·1	11·1	11·4	9·9	10·0

TABLE 8. CUMULATIVE SIZE DISTRIBUTIONS (PERCENTAGES) OBSERVED AND CALCULATED ($\pi = 0{\cdot}109$) (WYLLIE FED AT 5 G/MIN)

size (μ)	feed obs.	$\nu=1$ obs.	calc.	$\nu=2$ obs.	calc.	$\nu=3$ obs.	calc.	$\nu=4$ obs.	calc.	$\nu=5$ obs.	calc.	$\nu=10$ obs.	calc.
2411	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0
1801	61·0	63·5	64·8	68·3	68·2	72·1	71·3	76·1	74·1	79·2	76·6	88·0	85·9
1345	23·3	31·0	30·1	38·1	36·2	44·4	41·9	49·9	47·0	55·0	51·7	72·0	69·7
1005	0	9·0	7·8	16·3	15·0	23·4	21·6	28·9	27·8	34·8	33·4	54·6	56·0
751	0	6·6	6·3	13·0	12·2	18·1	17·8	23·0	23·1	28·0	28·2	46·6	49·3
561	0	5·4	4·9	10·3	9·8	15·3	14·5	19·5	19·0	23·4	23·4	40·4	43·0
419	0	4·4	3·9	8·6	7·8	12·8	11·6	16·5	15·5	19·9	19·3	35·0	37·1
313	0	3·6	3·0	7·1	6·1	10·7	9·3	13·6	12·5	16·6	15·7	30·6	31·2
234	0	3·1	2·3	6·0	4·7	9·1	7·3	11·8	10·0	14·3	12·8	26·4	27·2
175	0	2·7	1·7	5·2	3·7	7·8	5·8	10·0	8·0	12·0	10·3	22·9	23·1

Wyllie: high feed rate

In this experiment a 100 g sample of Wyllie coal was fed into the cone mill rotating at 125 rev/min, not at 5 or 10 g/min as before but at 100 g/min. It is not at present known exactly what effect this had on the motion of the balls in the mill or on the particles in the grinding zone, but that the type of breakage differs becomes immediately apparent. The distributions reported for one, two and three cycles in table 10 have a different form from those previously given, and cannot be described as the products of repeated cycles of π-breakage; those particles which are broken are crushed to a finer product than hitherto.

The assumption made to fit these distributions is that in each cycle a proportion π of the particles is broken, and the product of this breakage is immediately rebroken. The breakages are still to be according to our breakage function. The corresponding breakage matrix is

$$\pi \mathbf{B}^2 + (1-\pi)\,\mathbf{I}.$$

The previous theory applies to this process with a single change; $\mathbf{B}^2$ is to be substituted for $\mathbf{B}$.

On this assumption π is estimated by π_5 by a similar method to that above, and for the first five cycles we have the estimates of table 9, with arithmetic mean 0·071. The cumulative distributions observed and calculated are given in table 10. The agreement must still be considered good.

Table 9. Estimation of π (Wyllie fed at 100 g/min)

ν	=	1	1	3	4	5
$100\pi_5$	=	6·9	7·6	7·0	7·5	6·4

Table 10. Cumulative size distributions (percentages) observed and calculated (π = 0·071) (Wyllie fed at 100 g/min)

size (μ)	feed obs.	$\nu=1$ obs.	$\nu=1$ calc.	$\nu=2$ obs.	$\nu=2$ calc.	$\nu=3$ obs.	$\nu=3$ calc.
2411	100·0	100·0	100·0	100·0	100·0	100·0	100·0
1801	61·0	62·0	63·8	65·0	66·3	69·0	68·7
1345	21·5	27·0	27·0	35·0	32·0	42·0	36·7
1005	0	7·8	6·7	15·7	13·0	22·6	18·9
751	0	6·6	6·4	13·4	12·3	19·4	17·9
561	0	5·6	5·9	11·5	11·4	16·6	16·6
419	0	4·7	5·3	9·8	10·4	14·0	15·2
313	0	4·0	4·7	8·2	9·2	12·1	13·6
234	0	3·4	4·1	7·1	8·1	10·3	12·0
175	0	2·9	3·5	6·1	7·0	8·8	10·5

Discussion

We conclude from these results that under the conditions described, the treatment advanced in this paper is justified. The two assumptions made have now been tested (and are further tested in the three experiments described below). The calculation by matrix methods of a number of cycles of breakage is completed without difficulty; the agreement between calculations and experiments shows that the concept of selection, with the breakage function chosen, is realistic.

It is particularly striking that the calculated percentages undersize agree with the experimental results for the larger particle sizes, although in the estimation of π the percentages under the smaller particle sizes only were used. That is, the size distributions above 1 mm were ignored in the estimation, but are correctly represented by the calculations.

In the treatment of Bennett, Brown & Crone (1941) the larger particles would be termed residue, and it would be considered that the breakage forces had penetrated these particles incompletely. The size distribution of the smaller particles (in their term, the complement) would be well described by their law, but attempts to describe the residue (e.g. Taylor 1953) cannot be considered satisfactory.

In studying a cycle of π-breakage we do not distinguish between residue and complement. Larger particles in the product arise either from the complete escape of particles in the feed or, in the case of broken particles, from that proportion of the product which the breakage function predicts will be near the size of the original particles. The concept of π-breakage succeeds in replacing those of residue and complement and in representing both adequately.

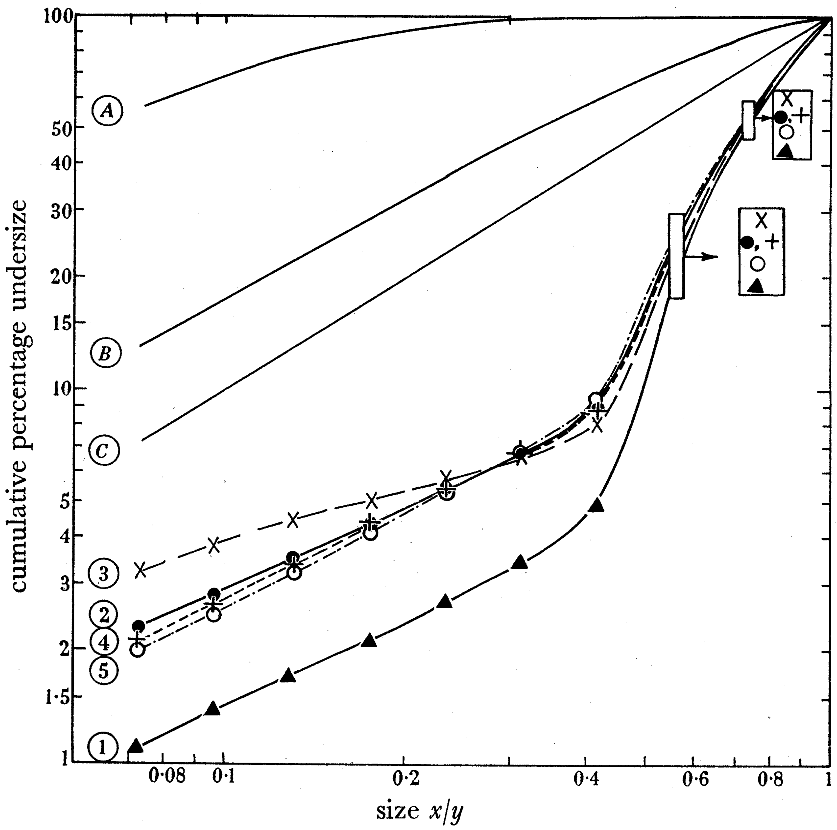


FIGURE 3. Comparison of three breakage functions. Products from breakage of single particles: A, $B(x/y) = 1-\exp(-10x/y)$; B, $B(x/y) = [1-\exp(-x/y)]/[1-\exp(-1)]$; C, $B(x/y) = x/y$. 1, feed to cone mill; 2, product from cone mill; 3, product calculated from A, $\pi = 3\cdot3\%$; 4, product calculated from B, $\pi = 6\cdot1\%$; 5, product calculated from C, $\pi = 7\cdot6\%$.

The experiment with high feed rate shows that cycles of π-breakage are not the only ones obtained with the cone mill. This suggested comparison of possible breakage functions with our experimental results. We accordingly fitted two other widely different functions to the breakage of Markham coal from the first to second cycles (table 6, $\nu = 1, 2$). The two functions were

(*a*) the 'straight-line' law: $b(x,y) = x/y$,

(*b*) the function $b(x,y) = 1-\exp(-10x/y)$.

The effect of breakage according to these functions and according to our breakage function on a single particle of size 1 is shown in figure 3. We show also the feed ($\nu = 1$) and product ($\nu = 2$) distributions actually obtained. Finally, π was estimated for each breakage function. For our function, as already stated, $\pi_5 = 6{\cdot}1\,\%$; for (*a*), $\pi_5 = 7{\cdot}6\,\%$, and for (*b*) $\pi_5 = 3{\cdot}3\,\%$. The predicted products are also shown on figure 3.

We conclude from this figure that the concept of selection for breakage is sufficiently elastic to accommodate these very different breakage functions with the experimental results. All three predicted products are of the same form as that actually found. At the same time, the products predicted by (*a*) and (*b*) are distinctly further from the actual product than that predicted by our function. It would be possible to choose particular breakage and selection functions to suit these circumstances better than ours do, but we consider the effort would be misguided. Our functions give good representations of other very different breakage processes.

Shatter test of coal friability

The friability of lump coal has been studied by dropping the coal from given heights on to a solid plane, often a steel plate, and assessing the breakage by means of the size distribution of the product. Brown (1948) used such tests to study both coal strength and the size distribution of 'complements'. Gilmore, Nicolls & Connell (1935) in applying a coke shatter test method (A.S.T.M. Standard D 141–23) to lump coal have given full size analyses of the product after each successive drop of two to three inch lumps. The complex analysis of the breakage may be simplified by examining the correspondence of each drop to a cycle of π-breakage.

For one series of tests, Gilmore *et al.* (1935, table 23, pp. 90–92) used 50 lb. samples of three coals, the coal being graded to pass a 3 in. screen and then hand-sized so that no lump appeared capable of passing in any position a 2 in. screen. Each sample was dropped one, two, three and four times. Except that a size analysis using square aperture screens was made after each drop, the technique corresponded to the coke shatter test standard.

We first assumed that the initial hand sizing was consistent with the screen analyses of the products. We used a 7×7 $\mathbf{B}$ matrix ($a = \frac{2}{3}$); hence $\mathbf{f} = \{1, 0, \ldots, 0\}$. The first column of $\mathbf{B}$ is

$$\{0{\cdot}1179,\ 0{\cdot}2128,\ 0{\cdot}1814,\ 0{\cdot}1411,\ 0{\cdot}1047,\ 0{\cdot}0751,\ 0{\cdot}0525\}$$

and $0{\cdot}1145$ undersize a^7.

The product vectors corresponding to the screen analyses are given in table 11. Table 12 contains the estimated π for each drop assuming that each drop corresponded to a cycle of π-breakage.

For Pennsylvania anthracite, the four estimates of π are nearly constant, averaging $0{\cdot}04$, whereas for the Nova Scotia and British Columbia bituminous coals π tended to decrease as the number of drops increased. This probability arose from the assemblies becoming stronger and the effective impact decreasing as the proportion of smaller sizes increased. Gilmore *et al.* considered that some additional tests showed shielding or cushioning by the smaller particles to have been unimportant.

Product vectors calculated from each cycle to the next are also given in table 11; for Pennsylvania anthracite the average π ($0{\cdot}04$) has been used in these calculations. The good general agreement between the observed and calculated product vectors suggests that each drop approximated to a cycle of π-breakage. The most serious discrepancies, which occur at the larger sizes after the first cycle of breakage, may result from the dubious assumption that the initial hand sizing and the screen analysis are consistent. We therefore ask, can the likely size distribution of the original coal in terms of screening be estimated from the data?

To answer this question we require an estimate of π in the first cycle of breakage. Only for Pennsylvania anthracite have we such an estimate; the average π over the last three

TABLE 11. CUMULATIVE SIZE DISTRIBUTIONS (PERCENTAGES) IN SHATTER TESTS, OBSERVED AND CALCULATED (DATA FROM GILMORE *ET AL.*)

Pennsylvania anthracite ($\pi = 0{\cdot}040$)

size (in.)	$\nu=1$ obs.	$\nu=1$ calc.	$\nu=2$ obs.	$\nu=2$ calc.	$\nu=3$ obs.	$\nu=3$ calc.	$\nu=4$ obs.	$\nu=4$ calc.
3	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0
2	12·4	3·5	20·8	15·5	23·7	23·6	29·0	26·4
1·347	4·2	2·7	7·4	6·8	10·4	10·0	13·0	12·9
0·924	2·4	2·0	4·5	4·4	6·6	6·5	8·3	8·6
0·605	1·4	1·4	2·7	2·9	4·2	4·2	5·5	5·7
0·405	0·9	1·0	1·9	1·9	2·9	3·0	3·9	4·0
0·272	0·6	0·7	1·4	1·3	2·1	2·2	2·9	2·9
0·182	0·5	0·5	1·0	1·0	1·5	1·6	2·2	2·1

Nova Scotia bituminous

size (in.)	$\nu=1$ obs.	$\nu=1$ calc.	$\nu=2$ obs.	$\nu=2$ calc.	$\nu=3$ obs.	$\nu=3$ calc.	$\nu=4$ obs.	$\nu=4$ calc.
	$\pi=0{\cdot}196$		$\pi=0{\cdot}126$		$\pi=0{\cdot}147$		$\pi=0{\cdot}095$	
3	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0
2	29·5	17·3	40·4	37·3	50·5	49·1	56·7	54·6
1·347	15·5	15·1	23·6	23·0	32·5	32·6	38·5	37·1
0·924	10·1	9·5	16·7	16·2	24·0	24·5	28·4	28·1
0·605	6·4	6·8	11·0	11·2	16·9	17·6	20·1	20·5
0·405	4·3	4·7	7·6	7·9	12·0	13·0	15·0	15·0
0·272	3·0	3·3	5·6	5·7	8·4	9·9	10·7	10·8
0·182	2·3	2·2	4·0	4·3	5·8	7·5	8·0	7·7

British Columbia (Nicola) bituminous

size (in.)	$\nu=1$ obs.	$\nu=1$ calc.	$\nu=2$ obs.	$\nu=2$ calc.	$\nu=3$ obs.	$\nu=3$ calc.	$\nu=4$ obs.	$\nu=4$ calc.
	$\pi=0{\cdot}296$		$\pi=0{\cdot}202$		$\pi=0{\cdot}100$		$\pi=0{\cdot}124$	
3	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0
2	36·8	26·1	56·1	48·1	63·9	60·0	70·4	67·9
1·347	25·6	19·8	40·0	36·1	45·5	44·4	53·5	50·5
0·924	16·5	14·4	28·0	25·9	33·6	32·3	41·0	38·6
0·605	9·4	10·3	17·4	17·4	22·2	21·5	26·5	27·1
0·405	5·7	7·2	11·0	12·1	13·5	14·5	17·5	18·0
0·272	3·6	4·9	6·8	8·5	8·6	9·7	11·4	12·4
0·182	2·2	3·4	4·6	5·9	5·8	6·9	7·8	8·8

TABLE 12*a*. BALL MILL TESTS: CUMULATIVE FEED DISTRIBUTION FOR ALL TESTS WITH SMALL BALL MILL (BILSTHORPE COAL)

	a^0	a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8
sizes (μ) ...	1204	872	632	457	331	240	174	126	91
percentage undersize a^i	100·0	48·0	2·7	nil	nil	nil	nil	nil	nil

TABLE 12*b*. SMALL BALL MILL TESTS USING 2960 G OF $\frac{1}{4}$ IN. BALLS AND 130 G OF COAL

duration of grinding	sizes ...	a^0	a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	π	sample no.
		cumulative percentage undersize a^i										
$2\frac{1}{2}$ min	obs.	100·0	65·0	26·6	14·7	10·9	8·2	6·4	4·9	4·0	0·236	1
	calc.	100·0	59·1	21·3	15·3	11·9	9·1	6·8	5·1	3·7		
5 min	obs.	100·0	70·0	36·3	22·2	18·0	14·3	11·5	9·3	7·4	0·146	2*
	calc.	100·0	69·6	35·4	23·3	18·1	14·1	11·1	8·6	6·9		
$7\frac{1}{2}$ min	obs.	100·0	74·0	43·7	30·0	24·0	20·6	16·5	14·0	11·7	0·146	3*
	calc.	100·0	74·0	44·0	30·1	24·8	20·0	16·2	13·2	10·6		
10 min	obs.	100·0	76·7	48·5	35·3	29·7	25·0	21·2	18·5	16·0	0·126	4*
	calc.	100·0	77·1	49·9	36·6	29·9	25·0	20·8	17·5	14·6		

* For these tests a loss of about 1 % of coal charge was included as coal smaller than a^8.

TABLE 12*c*. SMALL BALL MILL TESTS USING 2360 G OF $\frac{1}{4}$ IN. BALLS AND 130 G OF COAL

duration of grinding	sizes ...	a^0	a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	π	sample no.
		cumulative percentage undersize a^i										
$2\frac{1}{2}$ min	obs.	100·0	65·5	27·4	15·0	11·6	9·1	7·1	5·6	4·4	0·194	5*
	calc.	100·0	59·2	21·4	15·4	12·0	9·1	6·9	5·1	3·8		
5 min	obs.	100·0	70·5	36·9	22·5	18·9	15·6	12·8	10·7	9·3	0·155	6*
	calc.	100·0	70·3	36·6	24·1	19·1	15·2	12·0	9·5	7·5		

* Total losses were less than 0·4 % and were included as finer than a^8.

TABLE 12*d*. SMALL BALL MILL TESTS USING 2960 G OF $\frac{1}{4}$ IN. BALLS AND 64 G OF COAL

duration of grinding	sizes ...	a^0	a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	π	sample no.
		cumulative percentage undersize a^i										
$2\frac{1}{2}$ min	obs.	100·0	68·1	31·6	18·5	14·7	11·9	9·5	7·8	6·6	0·290	7*a**
	calc.	100·0	62·5	27·1	20·4	16·2	12·7	10·0	7·8	6·2		
5 min	obs.	100·0	70·4	40·0	25·2	23·7	22·0	19·5	16·9	14·0	0·191	7*b**
	calc.	100·0	74·0	43·1	30·2	24·7	20·3	16·6	13·7	11·5		

* Sample no. 7 was split into two equal parts; total losses were less than 0·4 % and were included as finer than a^8.

TABLE 12*e*. SMALL BALL MILL TESTS USING 2500 G OF $\frac{5}{8}$ IN. BALLS AND 130 G OF COAL

duration of grinding	sizes ...	a^0	a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	π	sample no.
		cumulative percentage undersize a^i										
$2\frac{1}{2}$ min	obs.	100·0	85·2	55·0	36·5	25·0	18·2	12·8	9·2	7·0	0·507	8*
	calc.	100·0	71·9	42·7	33·0	25·6	19·5	14·7	10·9	8·1		
5 min	obs.	100·0	94·5	77·6	58·8	44·2	33·1	23·8	17·6	13·3	0·405	9*
	calc.	100·0	90·6	70·4	55·6	44·3	35·7	29·5	23·7	19·3		
$7\frac{1}{2}$ min	obs.	100·0	97·8	90·1	76·8	62·5	48·5	36·0	26·7	20·2	0·376	10*
	calc.	100·0	96·4	84·9	71·0	58·9	48·4	38·8	31·3	25·3		

* Total losses were less than 0·4 % and were included as finer than a^8.

TABLE 12*f*. MEDIUM BALL MILL TESTS USING 1 IN. BALLS AND 15·3 KG OF COAL (CHISNALL HALL)

		cumulative percentage undersize a^i									
		a^0	a^1	a^2	a^3	a^4	a^5	a^6	a^7		mass of
sizes ...		3175*	1905	1143	656	412	247	148	89		sample (g)
feed ...		100·0	87·0	66·0	47·0	32·2	21·2	14·0	10·0		
duration of grinding										π	
$10\frac{1}{2}$ min	obs.	100·0	92·5	77·8	58·0	42·7	31·2	22·7	17·0	0·31	*ca.* 100†
	calc.	100·0	90·4	73·9	57·4	43·1	31·3	22·5	16·5		
21 min	obs.	100·0	97·0	88·0	70·0	54·5	39·8	28·0	20·5	0·31	*ca.* 100†
	calc.	100·0	94·5	83·1	66·9	52·8	41·0	31·4	23·6		
42 min	obs.	100·0	99·0	98·0	96·5	83·5	64·0	44·6	31·2	0·81	*ca.* 100†
	calc.	100·0	99·1	95·6	87·7	77·2	65·1	52·8	41·9		

* a^0 is taken as 0·125 in. or 3175μ.

† Losses on sieving were less than 0·2 % and were included as coal finer than a^7.

cycles. For the other two coals no estimate of π can be made which does not assume the feed distribution known. Using the estimated π, 0·040, we calculate the feed distribution of Pennsylvania anthracite from the product of the first cycle. We obtain, not the vector originally assumed, but

$$\{0·908,\ 0·077,\ 0·011,\ 0·004,\ 0·001,\ 0,\ 0\}.$$

This suggests that the initial hand sizing of the lumps gave as retained on a 2 in. screen nearly 10 % of the sample which would normally during sieving have passed it.

This examination of the results of Gilmore *et al.* has shown that these shatter tests can be simply described by estimating the value of π for each drop; but that particular attention should be paid to ensuring consistent techniques for sizing the lumps. Brown's (1948) work suggests that π provides an estimate of the strength of assemblies prior to each drop. With a wider verification of the model describing the breakage mechanism, it may be possible to simplify the interpretation of shatter tests.

Batch ball milling

The description of closed circuit ball-mill grinding will be simplified if it can be shown that periods of grinding correspond to cycles of π-breakage. It is not possible to define *a priori* the operating conditions for which this correspondence will take place. The examination of the crushed products from two smooth-walled cylindrical ball mills shows that, for certain operating conditions, the breakage mechanisms can be considered as repeated cycles of π-breakage, whereas other conditions of operation involve more complex processes.

The smaller of the two mills, of internal length 5·75 in. and diameter 3·85 in., was rotated by rollers at 74 rev/min. and charged with various weights of steel balls and with carefully prepared samples from a single batch of Bilsthorpe coal. Eight element vectors were used to describe the feed and product distributions, with $a = 0·7242$. The first column of **B** was therefore

$$\{0·0935,\ 0·1786,\ 0·1583,\ 0·1326,\ 0·1068,\ 0·0835,\ 0·0639,\ 0·0482\},$$

with 0·1346 undersize a^8. The feed vector is given in table 12*a*.

In the first test the mill was charged with 2960 g of $\frac{1}{4}$ in. diameter balls (52 % of mill volume including voids) and 130 g of coal. Separate samples were ground for $2\frac{1}{2}$, 5, $7\frac{1}{2}$ and 10 min; the product vectors are given in table 12*b*. The product of each $2\frac{1}{2}$ min grinding was assumed to be given by a cycle of π-breakage on the product of previous grinding. The values of π were estimated from (13) to be respectively 0·236, 0·146, 0·146 and 0·126. The product distributions calculated on this assumption are given in table 12*b*, and are in good agreement with the distributions observed.

In the second test only 2360 g of $\frac{1}{4}$ in. balls were charged (41 % of mill volume), but the conditions were otherwise similar to the first test. Successive periods of $2\frac{1}{2}$ min grinding were again equivalent to cycles of π-breakage, π being estimated as 0·194 and 0·155 for the first two cycles. The agreement between the distributions observed and calculated shown in table 12*c* is again good.

When the weight of coal charged to the mill was reduced to 64 g, the weight of $\frac{1}{4}$ in. balls being 2960 g, the first $2\frac{1}{2}$ min grinding was again approximated by a cycle of π-breakage; π was estimated as 0·290. The second $2\frac{1}{2}$ min grinding did not correspond to a cycle of π-breakage. The best fit to the product distribution, assuming π-breakage, is shown in table 12*d*, and differs in form from that observed. The actual breakage which occurred is not investigated here.

In the fourth test, the balls were of $\frac{5}{8}$ in. diameter; 2500 g of balls (50 % of mill volume) and 130 g of coal were charged to the mill. The breakage mechanism in this test differed drastically from π-breakage. The attempts to fit the observed product distributions, after $2\frac{1}{2}$, 5 and $7\frac{1}{2}$ min grinding, shown in table 12*e*, have failed completely. Under these conditions, the smaller particles must have had less probability of being broken than the larger. This situation would arise if the balls were unable to 'nip' the smaller particles. The results of table 12*e* contrast with those of tables 12*b* and 12*c*, in which the probability of breakage appears to be independent of particle size.

The larger of the two mills tested, of internal length and diameter 18 in., was rotated at 47 rev/min and charged with 15 % by volume of 1 in. diameter balls, and 15·3 kg of Chisnall Hall coal. Seven element vectors were used, with $a = 0{\cdot}6$. The first column of **B** was therefore

$$\{0{\cdot}1471,\ 0{\cdot}2648,\ 0{\cdot}2031,\ 0{\cdot}1413,\ 0{\cdot}0926,\ 0{\cdot}0585,\ 0{\cdot}0365\},$$

with 0·0561 undersize a^7.

Samples were taken from the central region of the mill after $10\frac{1}{2}$, 21 and 42 min grinding. The feed and product vectors observed are given in table 12*f*. We think that some size segregation may have existed at the sampling region, causing the smaller sizes to be under-represented in the sample. Nevertheless, the product of $10\frac{1}{2}$ min grinding is in reasonable agreement with that calculated from a cycle of π-breakage ($\pi = 0{\cdot}31$). The attempts to fit the observed distributions, at 21 and 42 min, by cycles of π-breakage (table 12*f*), have failed completely. More complex selection processes are at work, which are not further investigated here.

We conclude that the breakage mechanisms of a ball mill, under certain operating conditions, may be considered as repeated cycles of π-breakage, but our tests show the likelihood of more complex selection processes at least equals that of π-breakage.

Beater mill

Many laboratory and industrial machines impact coal particles with a rotating set of hammers. An individual particle may or may not be struck by a hammer when fed into the mill, and the product of breakage may or may not be rebroken against the mill wall or by other hammers. Analysis of these selection processes is very complex, since a large number of factors influence the selection. It is reasonable to expect the air currents set up by rapidly rotating hammers to favour the escape of the smaller particles. The laboratory mill whose characteristics are now studied was built in order to study the breakage resulting from the hammers alone.

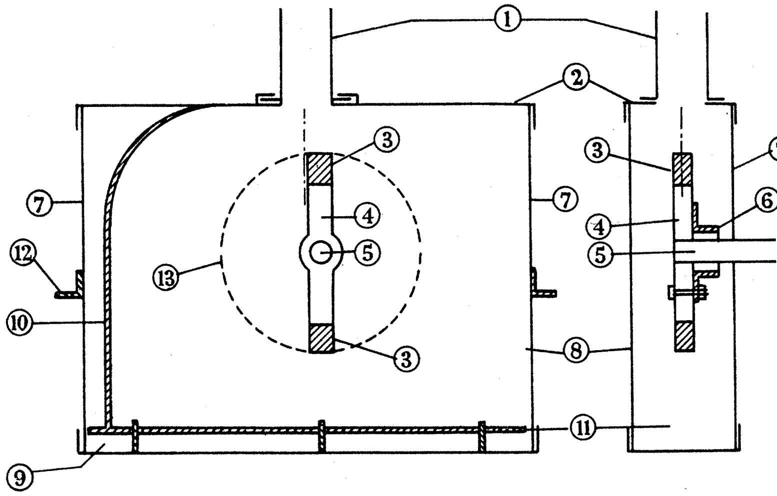


FIGURE 4. Laboratory beater mill. 1, feed chute, adjustable in slides; 2, tin lid, firm fitting with overlap and rubber gaskets; 3, steel sheath of dual beaters; 4, wooden core of beaters; 5, rotor from 1440 rev/min $\frac{1}{4}$ h.p. motor; 6, flanged plate, adjustable; 7, tin back and side walls; 8, Perspex front wall; 9, tin collector tray; 10, rubber arrestor on forward wall; 11, rubber arrestor in collector tray; 12, angle struts to position casing; 13, path and direction of rotation of beater tips.

The mill is shown in figure 4, the beaters were made of wood and sheathed in steel over a length sufficient to ensure that impacts occurred on the steel sheaths only. The feed chute was arranged so that particles fell, initially at least, along the centre line in figure 4. The walls and floor were covered with Sorbo rubber, arranged so that the kinetic energy of particles reaching them was readily dissipated without detectable breakage. During testing, particles were fed singly into the mill; the air currents set up by the beaters (1440 rev/min) tended to circulate small particles within the casing. Some small particles escaped through the joints of the casing, but careful operation kept these losses small.

Some tests with closely sized lead shot gave the proportions of each size struck by the hammers. The average diameters of the lead shot were 3·65, 2·76, 2·13 and 1·18 mm. Single particles of shot were dropped down the chute into the path of the hammers which were coated with a thin even layer of wax, and if the sphere was struck, a clean indentation could be seen. The proportions of hits for each size are given in figure 5. It does not seem possible to deduce quantitatively from these data the proportion of breakage for each size of coal particle; but direct estimation of the selection for breakage of coal particles from the feed and product distributions of Markham coal has been made, and the chance of breaking a particle diminished more rapidly with size than did the chance of striking the lead shot.

To describe the breakage we used a 7×7 **B** matrix with $a = 0{\cdot}6$; its first column was

$$\{0{\cdot}1471,\ 0{\cdot}2648,\ 0{\cdot}2031,\ 0{\cdot}1413,\ 0{\cdot}0926,\ 0{\cdot}0585,\ 0{\cdot}0365\}$$

with a fraction 0·0561 less than a^7.

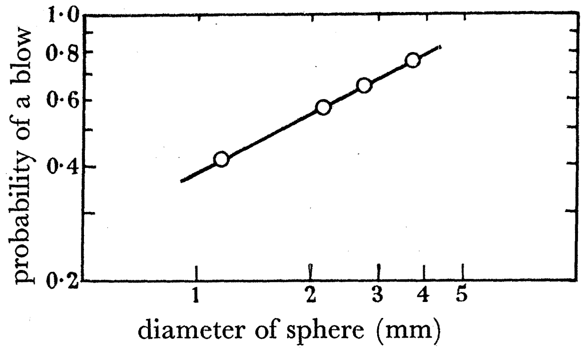


FIGURE 5. Probability of blows and particle size (70 spheres were dropped to obtain each point).

We considered that particles in the largest size range were certain to be broken, and that smaller particles had a rapidly diminishing chance of breakage. We therefore postulated the selection matrix with diagonal elements

$$1,\ \pi,\ \omega,\ 0,\ \ldots,\ 0,$$

where π and ω were to be estimated from the data of table 13. The necessary calculations were outlined in § 2. From equations similar to (11) which minimized the sum of squares of differences in the cumulative vectors, we calculated

$$\pi = 0{\cdot}911 \quad \text{and} \quad \omega = 0{\cdot}491.$$

The product calculated using **B** and this selection matrix is given in table 13. The agreement with the observed product is very good.

We conclude that for these coal particles the probability of breakage in this mill falls rapidly to zero, being about 1, 0·911 and 0·491 for the first three size ranges.

TABLE 13. CUMULATIVE DISTRIBUTION IN BEATER MILL (MARKHAM COAL)

	a^0	a^1	a^2	a^3	a^4	a^5	a^6	a^7
size (mm) ...	12·7	7·62	4·57	2·74	1·65	0·99	0·598	0·358
feed	100·0	78·0	53·5	35·5	20·0	11·7	6·7	3·8
product								
obs.	100·0	97·0	85·5	64·5	41·0	23·0	13·0	7·1
calc.	100·0	96·8	85·5	64·6	39·1	23·9	14·2	8·4

Conclusion

It has been shown that several machines effect a breakage very similar to π-breakage, the simplest selection process we have considered. In some cases the breakage was more complex; it is then necessary to discover a more realistic selection process. There are two starting-points for this search; our knowledge of the construction of the machine considered, and the particular discrepancies between its product and that predicted by π-breakage (for example, the discrepancies will usually show whether the larger particles have a higher probability of breakage than the smaller or vice versa). The selection function which gives an adequate representation of the process directs attention to the actual operation of the machine, and may point the way to improvements in its efficiency.

We have approximated to size distributions by vectors, and this initial simplification has enabled us to manipulate distributions easily and without further approximation. In vector form, the essential features of a distribution are quickly grasped. The flexibility of the matrix treatment promises a wide range of applications.

In the study of coal breakage, the use of selection and breakage functions may reduce a mass of data to a few simple principles and parameters, and parameters such as π may summarize much useful information.

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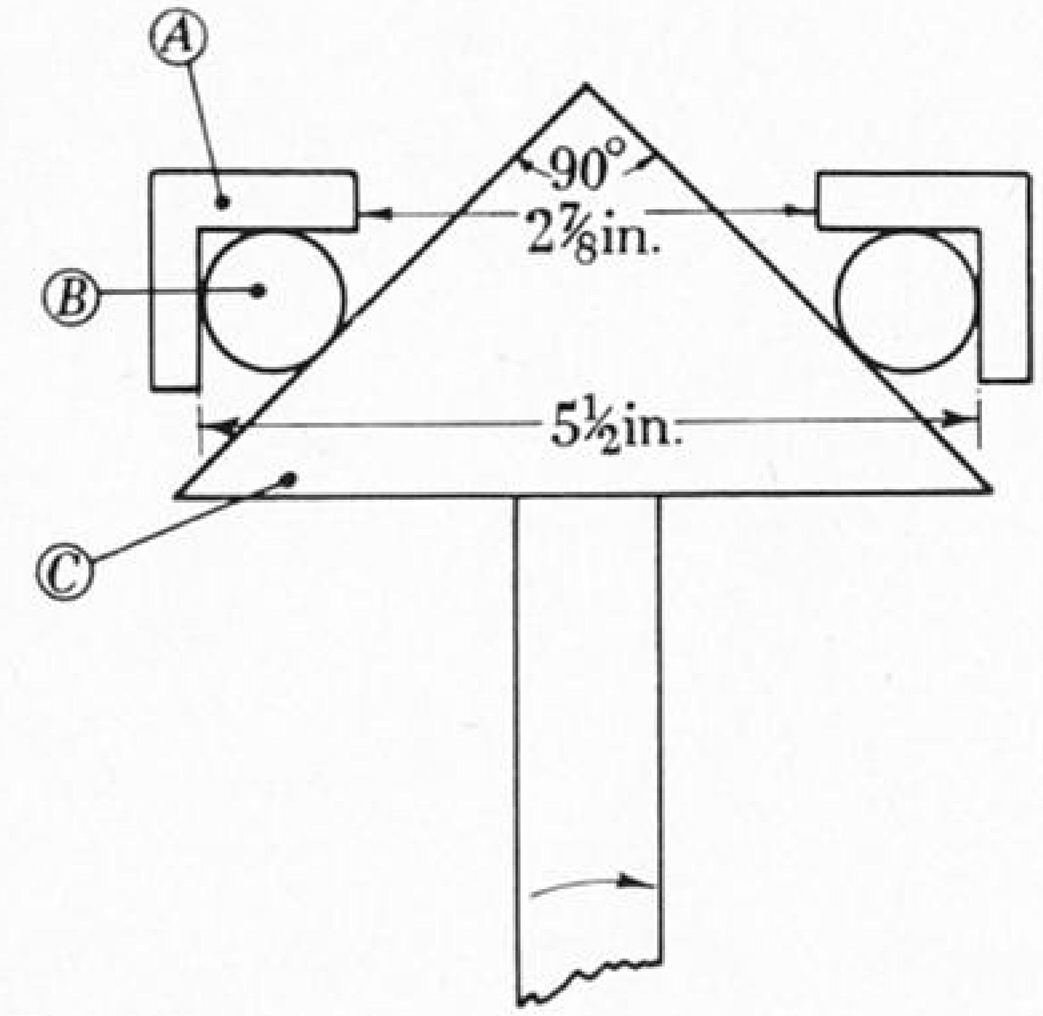
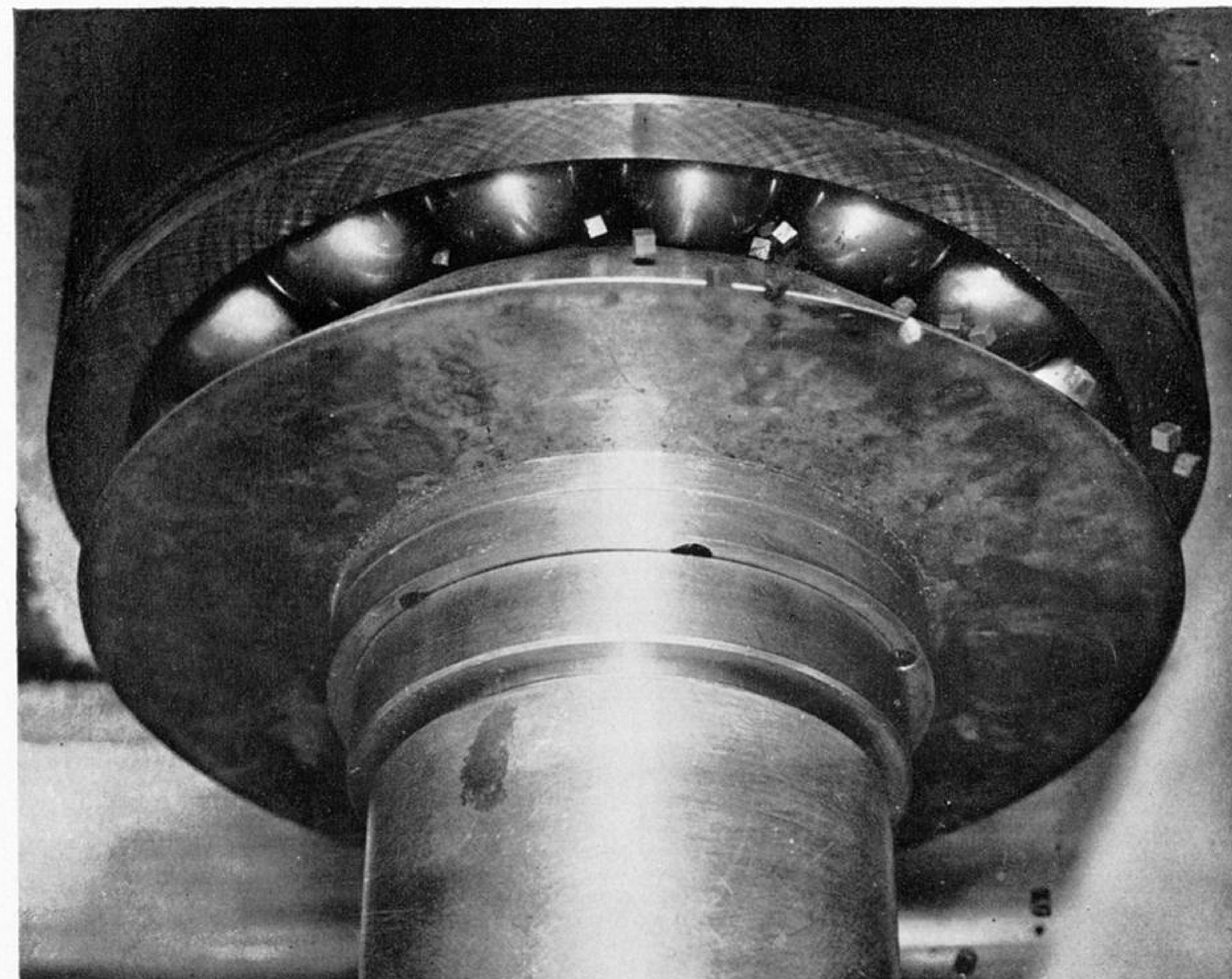


(b)

(a)



(c)

FIGURE 1. (a) General view of cone mill with cover lowered. (b) Section through cone mill. (c) High-speed flash photograph of $\frac{1}{4}$ in. cubes of coal passing through the grinding elements of the cone mill which rotated at 125 rev/min.